MINKOWSKI DECOMPOSITION OF CONVEX SETS

BY

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ABSTRACT

A set K is decomposable if it can be written as the Minkowski sum $A + B$ where neither A nor B is homothetic to K. In this paper, it is shown that a wide class of convex sets is decomposable including those which contain a sufficiently smooth neighborhood on their boundary.

1. Introduction

Given two closed convex sets A, B contained in E^d, we may define their *Minkowski sum* or *vector sum* $A + B$ as $\{x: x = a + b,\ a \in A,\ b \in B\}$. Conversely, given any closed convex set K, the following question may be posed: For what A and B is $K = A + B$? It may happen that whenever $K = A + B$, A and B are *homothets* of K, that is, of the form $\lambda K + x_0$ (where $\lambda K = \{\lambda x: x \in K\}$) for some $\lambda > 0$ and some $x_0 \in E^d$. In such a case we say K is *indecomposable*. Otherwise, we say K is *decomposable*.

The problem of decomposing convex sets has received the attention of various authors [3], [5], [8], [9]. A good summary of known results may be found in [4, §15]. In general, the decomposability of convex sets in E^2 is completely determined (triangles and line segments are the only indecomposable sets) and not a great deal is known, except for polytopes, if $d \geqq 3$.

It is the purpose of this paper to show that a wide class of convex sets is decomposable, the only condition on the sets being that they have on their surface a neighborhood which is sufficiently "nice". In order to precisely describe this notion, we recall the definition of the *ε-inner parallel body*, K_ε of a convex set K;

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$K_\varepsilon = \{x \in K : x + \varepsilon B \subseteq K\}$ where B is the unit ball. Now let $L_\varepsilon = K_\varepsilon + \varepsilon B$. Clearly, $L_\varepsilon \subseteq K$, but in general, $L_\varepsilon \neq K$, because it excludes the vertices of K. For example, if K is a unit square, K_ε is a square with sides parallel to K but a distance ε away, and L_ε is all of K except that the corners are rounded off. We will say that a neighborhood N contained in the boundary of K is *ε-smooth* if $N \subseteq L_\varepsilon$. Intuitively speaking, this means that a ball of radius ε can be moved around inside of K to touch every point of N. We can now state our principal result.

(1.1) THEOREM. *Suppose K is a closed, convex set in E^d whose boundary contains a neighborhood U which is rotund (that is, contains no line segments) and ε-smooth for some $\varepsilon > 0$. Then K is decomposable.*

(1.2) COROLLARY. *Suppose K is a closed, convex set in E^d whose boundary contains a neighborhood U which is rotund and which is twice continuously differentiable. Then K is decomposable.*

The idea of the proof is to take two copies of K and modify them so that the surface on one of them is slightly "more convex" in U while the other one becomes slightly "less convex" in U. If we do this carefully, the resulting sets A', B' are convex and $(\frac{1}{2}A') + (\frac{1}{2}B') = K$.

For various reasons it turns out to be somewhat easier to work with the *support function* of K, $h(K,u) = \sup\{\langle x,u\rangle : x \in K\}$, where $\langle \cdot,\cdot \rangle$ denotes the usual inner product. We will examine support functions in some detail in section 2. Section 3 is devoted to technical lemmas on convex functions, while section 4 is concerned with decomposing balls in a particular way. The proof of our main theorem is found in section 5, and a generalization is stated in section 6. The proof of the corollary is given in section 7 and concluding remarks in section 8.

2. Convex functions

Support functions have many useful properties which are examined in detail in, for example, [2] and [10]. We need only the following:

(2.1) *A closed convex set $K \subseteq E^d$ is completely determined by its support function $h(K,u)$.*

(2.2) *If A and B are closed convex sets in E^d, then*

$$h(A + B, \mathbf{u}) = h(A, \mathbf{u}) + h(B, \mathbf{u}).$$

(2.3) $$h(K, \lambda \mathbf{u}) = \lambda h(K, \mathbf{u}) \textit{ for all } \lambda \geqq 0.$$

(2.4) $$h(K, \lambda \mathbf{u} + (1-\lambda)\mathbf{u}) \leqq \lambda h(K, \mathbf{u}) + (1-\lambda) h(K, \mathbf{v}) \textit{ for all } 0 \leqq \lambda \leqq 1.$$

The properties stated in (2.3) and (2.4) are usually expressed by saying $h(K,\mathbf{u})$ is *positively homogeneous* and *convex*, respectively. It turns out that these two properties characterize support functions [2, p. 26].

(2.5) *Let h be any positively homogeneous, convex (real-valued) function defined on E^d. Then h is the support function of some convex set K.*

Thus, in the light of the results quoted above, decomposing a set K is equivalent to writing one positively homogeneous convex function as the sum of two others and this is precisely what we intend to do. In fact, we wish to make use of an idea, apparently first introduced in [9], to reduce the problem to that of writing one convex function as the sum of two others.

If f is any positively homogeneous function defined on $H^+ = \{x \in E^d : x = (x_1,\cdots,x_d),\ x_1 > 0\}$, we define f^1 as the restriction of f to $H^1 = \{x \in E^d : x_1 = 1\}$.

LEMMA. *Suppose f, f^1, H^+ and H^1 are as defined above. Then f is convex over H^+ if and only if f^1 is convex over H^1.*

PROOF. It is clear that f^1 is convex if f is, being a restriction of f. Conversely, suppose f^1 is convex. Then we may write

$$f(x) = x_1 f^1(x_2/x_1,\cdots,x_d/x_1).$$

Now suppose $x,\ y \in H^+$. It suffices to show that

$$\tfrac{1}{2}[f(x) + f(y)] \geqq f(\tfrac{1}{2}x + \tfrac{1}{2}y) = \tfrac{1}{2}f(x+y)$$

where the last equality follows by the positive homogeneity of f. Now

$$f(x) + f(y) = x_1 f^1(x_2/x_1,\cdots,x_d/x_1) + y_1 f^1(y_2/y_1,\cdots,y_d/y_1)$$

$$= (x_1 + y_1)\left[\left(\frac{x_1}{x_1+y_1}\right)f^1(x_2/x_1,\cdots,x_d/x_1) + \left(\frac{y_1}{x_1+y_1}\right)f^1(y_2/y_1,\cdots,y/y_1)\right]$$

$$\geqq (x_1+y_1)f^1((x_2+y_2)/(x_1+y_1),\cdots,(x+y_d)/(x_1+y_1)) = f(x+y).$$

Multiplying by $\frac{1}{2}$ gives the result.

Thus if h is a support function and we can write $2h^1$ as the sum of two other convex functions f^1 and g^1 on H^1, with the additional restriction that $f^1 = h^1 = g^1$ outside of some bounded set, then our problem is solved, because the final restriction will guarantee that f and g are support functions by our next result. More formally, we wish to establish

PROPOSITION (2.7). *Suppose that h is a support function on E^d which is positive except at the origin and that N is a bounded subset of H^1. Suppose, moreover,* *that f is a positively homogeneous function on E^d such that f^1 is convex on H^1 and such that $f(\mathbf{x}) = h(\mathbf{x})$ if $\mathbf{x} \notin \text{pos}(N)$. Then f is a support function on E^d.*

In the statement of the proposition above pos(N) denotes the *positive hull* of $N = \{\mathbf{x} \in E^d : \mathbf{x} = \lambda \mathbf{y}$ for some $\mathbf{y} \in N$ and some $\lambda \geqq 0\}$. To establish the assertion, some preliminaries are needed.

As in [7] we say that if f is any real-valued function on E^d the *epigraph of F*, $\text{epi}(f) = \{(\mathbf{x}, \mu) : \mathbf{x} \in E^d, \mu \in E^1, f(\mathbf{x}) \leqq \mu\}$. Clearly $\text{epi}(f) \subseteq E^d \times E^1$. The following fact about epigraphs will be useful [7, p. 25].

(2.8) *f is a convex function on E^d if and only if* epi(f) *is a convex subset of $E^d \times E^1$.*

(2.9) LEMMA. *Suppose f is a real-valued function of a real variable such that f is convex on $[a,c]$, f is convex on $[b,d]$ and $a \leqq b < c \leqq d$. Then f is a convex function on $[a,d]$.*

PROOF. Let $f'_+(x)$ denote the right derivative function on f on the open interval (a,d). By [7, §24], f is convex on $[a,d]$ if and only if f'_+ is a monotone increasing function. By assumption f'_+ is monotone on (a,c) and on (b,d). Since $b<c$. it follows that f'_+ is monotone on (a,d) and the proof is complete.

PROOF OF (2.7). Since f is a positively homogeneous function it suffices to show that it is convex. By (2.6), f is convex on H^+, and by assumption, f is convex on $E^d \sim \text{pos}(N)$ [as usual, $A \sim B = \{x : x \in A, x \notin B\}$]. Moreover, since f is positive except at the origin and positively homogeneous, f is convex on every line through the origin.

Let $[\mathbf{x}, \mathbf{y}]$ be any line segment which misses the origin. Then we can write

$$[\mathbf{x},\mathbf{y}] = ([\mathbf{x},\mathbf{y}] \cap H^+) \cup ([\mathbf{x},\mathbf{y}] \cap (E^d \sim \text{pos}(N)))$$

where the two subsets overlap since N is bounded.

By (2.9) the convexity of f on each part implies the convexity of f on all of $[\mathbf{x},\mathbf{y}]$. Hence f is convex over every line segment and thus convex over all of E^d.

3. Upper and lower convex envelopes

From the work of the preceding section, we see that if $h(K, \ )$ is the support function of K it suffices to write $2h^1(K, \ )$ as the sum of two convex functions f^1 and g^1, where $f^1 = g^1 = h^1(K, \ )$ outside a given bounded neighborhood N of H^1. Here we examine what restrictions this places on f^1 and g^1. Since H^1 is, for all geometrical purposes, E^{d-1}, we will phrase our results in terms of functions on Euclidean spaces.

Given any convex function f (not necessarily positively homogeneous) defined on E^d and any bounded convex set $N \subseteq E^d$, we associate two other convex functions $f^*(N;\ )$ and $f_*(N;\ )$, which we term respectively the *upper* and *lower convex envelopes of f with respect to N*. Intuitively $f^*(N;\ )$ is the largest convex function which agrees with f outside of N while $f_*(N;\ )$ is the smallest one.

Let $G(N,f)$ denote the set of all convex functions g in E^d such that $g(\mathbf{x}) = f(\mathbf{x})$ for all $\mathbf{x} \notin N$. Then we define

$$f^*(N;\mathbf{x}) = \sup\{g(\mathbf{x}) : g \in G(N,f)\}.$$

Since $f^*(N;\ )$ is the pointwise supremum of a collection of convex functions, it too is convex [7, p. 35]. Note that $f^*(N,\mathbf{x}) = f(\mathbf{x})$ for all $\mathbf{x} \notin N$.

The definition of $f_*(N;\ )$ is somewhat more complicated. Let N' be the set of points at which f is finite. We require that the interior of N' contain the closure of N for $f_*(N;\ )$ to be defined. Let $\mathscr{H}(N,f)$ denote the set of all those halfspaces of $E^d \times E^1$ which contain epi(f) and which have a bounding hyperplane supporting epi(f) at some point $(\mathbf{x},f(\mathbf{x}))$ with $\mathbf{x} \in N' \sim N$. Then $R = \cap\{H : \mathscr{H} \in K(N,f)\}$ is a non-empty convex set and hence [7, p. 23] the epigraph of a convex function which we define to be $f_*(N;\ )$.

(3.1) *Suppose f and g are convex functions on E^d, N a convex set and $h = f + g$. Then*

$$h^*(N;\mathbf{x}) \geqq f^*(N;\mathbf{x}) + g^*(N;\mathbf{x}).$$

PROOF. Since $f^*(N;\ ) \in G(N,f)$ and $g^*(N;\ ) \in G(N,g)$, $f^*(N;\ ) + g^*(N;\ ) \in G(N,h)$. The result then follows immediately from the definition.

We wish to establish a similar relationship for $h_*(N;\ )$, but it is more difficult. We begin be establishing another characterization of $f_*(N;\ )$

(3.2) $$f_*(N;\mathbf{x}) = \inf\{g(\mathbf{x}) : g \in G(N,f)\}.$$

PROOF. If $g \in G(N,f)$, then epi$(g) \subseteq H$ for all $H \in \mathscr{H}(N,f)$. Hence epi$(g) \subseteq R$ and so $f_*(N;\mathbf{x}) \leqq g(\mathbf{x})$. We claim that $f_*(N;\mathbf{x}) = g(\mathbf{x}) = f(\mathbf{x})$ if $\mathbf{x} \notin N$ and so, in fact, $f_*(N;\ ) \in G(N,f)$. The assertion is then immediate.

Suppose that for some $\mathbf{x} \notin N$, $f_*(N,\mathbf{x}) < f(\mathbf{x})$. Then the hyperplane supporting epi(f) at $(\mathbf{x},f(\mathbf{x}))$ has $f_*(N;\mathbf{x})$ in the open halfspace not meeting epi(f). The only way this could fail to happen would be if f became infinite at $\mathbf{x}$ and the supporting hyperplane is vertical—contrary to assumption of the finiteness of $f(\boldsymbol{x})$. So if f is finite, $(\boldsymbol{x}, f_*(N; \boldsymbol{x})) \notin R$, contrary to the definition of $f_*(N;\ )$, and the proof is complete.

(3.3) *Suppose f and g are convex functions on E^d, N a convex set, and $h = f + g$. Then $h_*(N; \boldsymbol{x}) \leqq f_*(N; \boldsymbol{x}) + g_*(N; \boldsymbol{x})$.*

PROOF. Since $f_*(N; \boldsymbol{x}) \in G(N, f)$ and $g_*(N; \boldsymbol{x}) \in G(N, g)$, it follows that $f_*(N; \boldsymbol{x}) + g_*(N; \boldsymbol{x}) \in G(N, h)$. The result follows immediately from (3.2).

We remark in passing that it can be shown that equality holds in the result above.

(3.4) LEMMA. *Suppose f and g are functions on the real line which are equal outside of a closed bounded interval $[a, b]$. Suppose, moreover, that $f \leqq g$, that g is convex on $[a, b]$ and that f is convex everywhere. Then g is convex everywhere.*

PROOF. One again we use the fact that a real-valued function $p(x)$ is convex if and only if its right derivative function, $p'_+(x)$, [or its left derivative function, $p'_-(x)$] is monotone [7, §24]. By assumption, we deduce that $g'_+(x)$ exists everywhere and is monotone on each of the intervals $(-\infty, a)$, $[a, b)$, and $[b, \infty)$ separately. Likewise, $g'_-(x)'$ exists everywhere and is monotone on $(-\infty, a]$, $(a, b]$ and (b, ∞).

Now for $z < a$, $g'_+(z) = f'_+(z) \leqq f'_+(a) \leqq g'_+(a)$. The first inequality derives from the convexity of f, while the second follows from the fact that $f \leqq g$ and $f(a) = g(a)$. Thus $g'_+(x)$ is monotone on $(-\infty, b)$, and so g is convex on that interval.

We likewise have for $b < z$, $g'_-(b) \leqq f'_-(b) \leqq f'_-(z) \leqq g'_-(z)$. Thus $g'_-(x)$ is monotone on (a, ∞) and hence g is convex on this interval. By (2.9) g is convex everywhere.

(3.5) COROLLARY. *Suppose f is a convex function on E^d, N a bounded convex set, g a function equal to f outside of N and which satisfies $f_*(N; \boldsymbol{x}) \leqq g(\boldsymbol{x})$ for all $x \in E^d$. Then g is a convex function on E^d if and only if it is a convex function on N.*

PROOF. In one direction the assertion follows easily. To see it in the other direction, consider the restriction of $f_*(N;\ )$, and g to the line determined by $\boldsymbol{x}$, $\boldsymbol{y}$ for any $\boldsymbol{x}$, $\boldsymbol{y}$ in E^d. By (3.4) g is convex on this line. Since g is convex on every line, g is convex on all E^d.

4. Decomposing balls

Here we show how to decompose balls in a particular way which we will use for our more general decomposition.

(4.1) LEMMA. *Let h_B be the support function (restricted to H^1) of the unit ball, and let T be a bounded ball in H^1 centered at $(1,0,\cdots)$. Then we can write $2h_B = p + q$ where p and q are convex functions, rotationally symmetric about the origin, which are different from h_B, but which agree with h_B outside of T, and for which $h_{B*}(T;x) \leqq p(x) \leqq h_B(x) \leqq q(x)$. Moreover, there are infinitely many ways to do this.*

PROOF. By the rotational symmetry, it suffices to prove the assertion on any line in H^1 through the center of T and then complete the resulting function by symmetry. So we suppose h_B is defined on the real line with $T = (-\varepsilon, \varepsilon)$.

In this special case it is easy to verify that $h_B(x) = \sqrt{1 + x^2}$. Thus,

$$h_{B^*}(T;x) = \begin{cases} (1 + \varepsilon|x|)/\sqrt{1+\varepsilon^2} & \text{if } x \in T \\ h(x) & \text{if } x \notin T. \end{cases}$$

The equation for $h_{B^*}(T;x)$ is derived by observing that the tangent lines to $h_B(x)$ at the boundary of T determine $h_{B^*}(T;x)$. Taking the convex function $q(x)$ whose epigraph is the convex hull of epi$(2h_B - h_{B^*}(T;\ ))$ gives

$$q(x) = \begin{cases} h_B(x) & \text{if } x \notin T \\ \beta & \text{if } |x| \leqq \varepsilon/(4+3\varepsilon^2) \\ 2h_B(x) - h_{B^*}(T;x) & \text{otherwise} \end{cases}$$

where $\beta = (\sqrt{4+3\varepsilon^2} - 1)/\sqrt{1+\varepsilon^2}$. It is easy to see that q is convex and that $h_B(x) \leqq q(x)$.

If we define $p(x) = 2h_B(x) - q(x)$, then it is also easy to verify that $p(x)$ is convex and that $h_{B^*}(T;x) \leqq p(x) \leqq h_B(x)$. Moreover, if $0 < \lambda < 1$ the function $p_\lambda(x) = \lambda p(x) + (1-\lambda)h_B(x)$ and $q_\lambda(x) = 2h_B(x) - p_\lambda(x)$ also satisfy the conclusions of the proposition (as stated on the real line) and are not multiples of $p(x)$ or $q(x)$. Rotating the functions $p(x)$ and $q(x)$ in H^1 about the center of T and using (2.7) establishes the result in general.

5. Proof of main theorem

Note that a body K which is ε-smooth at a point $x \in bd(K)$ has a unique supporting direction $u(x)$ there. So each $x \in U$ corresponds to exactly one $u(x)$ in $U' = \{u(x): x \in N\}$. Conversely, since U contains no line segments, each $u(x) \in U'$ corresponds to exactly one $x \in U$. Since the correspondence is also bicontinuous, U and U' are homeomorphic.

We assign coordinates so that the point $(1, 0, \cdots, 0)$ of S^{d-1} is an interior point of U' and let H^1 be as usual.

Let $N = \mathrm{pos}(U') \cap H^1$. Then N is homeomorphic to U' and thus to U. In addition, we may choose a small open ball $M \subseteq N$ such that M is centered at $(1, 0, \cdots, 0)$ and so that the closure of M is contained in the interior of N.

The assumption that U is ε-smooth is equivalent to asserting that the function $j = h - \varepsilon h_B$ is convex over N where h is the support function of K and h_B is the support function of the unit ball—each support function being restricted to H^1. Note that by a proper choice of origin, we can guarantee that $h(x) > 0$ if $x \neq 0$. By (4.1) we can write $2h_B = p + q$, where $p = q = h_B$ outside of M and $h_{B*}(M;\ ) \leqq p$ everywhere. Now define

$$d(x) = \begin{cases} h(x) & \text{if } x \notin M \\ j(x) + \varepsilon p(x) & \text{if } x \in M \end{cases}$$

and

$$e(x) = \begin{cases} h(x) & \text{if } x \notin M \\ j(x) + \varepsilon q(x) & \text{if } x \in M. \end{cases}$$

We observe that $d + e = 2h$ and that both d and e are convex functions over M (being the sum of convex functions) and are convex outside of M (being equal to h). So by (3.5), d and e will be convex everywhere if we can establish $h_*(M; x) \leqq d(x)$ and a similar statement for $e(x)$.

Now for $x \in M$, $d(x) = j(x) + \varepsilon p(x) \geqq j_*(M; x) + \varepsilon h_{B*}(M; x) \geqq (j + \varepsilon h_B)_*(M; x) = h_*(M; x)$. The first inequality follows from the facts that $f \geqq f_*$ everywhere and that $p(x) = h_B(x)$ if $x \notin M$. The second inequality is proved in (3.4).

It is even easier to see that for $x \in M$,

$$e(x) = j(x) + \varepsilon q(x) \geqq j(x) + \varepsilon h_B(x) = h(x) \geqq h_*(M; x).$$

So by (3.5), d and e are convex functions. It only remains to show $d \neq h \neq e$. Equality might occur for our choice of p and q in (4.1). But in this case we can choose two other functions p', q' and define new functions d', e', where $d' \neq h \neq e'$. The arguments above then show that d' and e' are the desired functions on H^1.

6. A slight generalization

The hypotheses of our basic theorem can be weakened somewhat to permit the size of the interior tangent ball to vary with the point in question.

(6.1) THEOREM. *Suppose K is a closed, convex set in E^d whose boundary contains a neighborhood U which is rotund and such that $U \subseteq L = \bigcap \{L_\varepsilon : \varepsilon > 0\}$. Then K is decomposable.*

PROOF. Let $A_\varepsilon = L_\varepsilon \cap bd\,K$. Then each A_ε is a closed subset of the boundary of K, and we also have $U \subseteq \bigcap \{A_\varepsilon : \varepsilon > 0\}$. Since $\delta < \varepsilon$ implies $A_\varepsilon \subseteq A_\delta$, we can also write $U \subseteq \bigcap \{A_{1/n} : n \text{ an integer}\}$. Now U is of second category and it must follow that one of the $A_{1/n}$, say $A_{1/N}$, is also of second category and thus contains interior points [6, p. 68]. Let V be the corresponding neighborhood of interior points. Then the result follows by applying (1.1) to V with $\varepsilon = 1/N$.

7. Proof of corollary

By assumption, U contains no line segments. Hence, it suffices to show that there exists a subneighborhood $V \subseteq U$ such that $V \subseteq L_\varepsilon = K_\varepsilon + \varepsilon B$ for some $\varepsilon > 0$. Let W be any compact subneighborhood of U and let x lie in the interior of W. Since U is smooth, there is a unique supporting hyperplane H to K at x. Let $T(x, \delta)$ denote the ball of radius δ which meets the interior of K and is tangent to H at x. We first wish to show that for some $\delta > 0$, $T(x,\delta) \cap W = \{x\}$.

Let z be any other point of W. Without loss of generality we may assign coordinates so that K lies in the half-space $\{y : y_n \geq 0\}$, so that $x = 0$ and so that $z = (z_1, 0, \cdots, 0, z_n)$. Note that the hyperplane $H = \{y : y_n = 0\}$ is the supporting hyperplane to K at x. With these coordinates we may write any point $w \in W$ as $w = (w_1, \cdots, w_{n-1}, r(w_1, \cdots, w_{n-1}))$ for some real-valued function r (if it happens that W is so large that two points of W lie above the single point of H consider a smaller neighborhood W').

By our hypotheses the mixed partial derivatives $D_{ij}r(w)$ exist and are continuous throughout U, or more properly, throughout U', the projection of U onto H. Thus each $D_{ij}r(w)$ attains an upper bound α_{ij} on W.

Let $\alpha = \max(\alpha_{ij})$. The convexity of K guarantees that $\alpha > 0$. It is now an elementary matter to verify that for any direction v, $D_v(D_v r(w))$ $\{D_v r(w)$ denotes here the directional derivative of $r(w)$ in direction $v\}$ exists and is less than or equal to α [1, p. 109]. Thus the argument which we apply below to z is equally valid for any other point in W.

Let $\delta = \alpha/5$ and consider the ball $T(x,\delta)$. If $|z_1| > \delta$, it is clear that $z \notin T(x,\delta)$, so we may assume $|z_1| \leqq \delta$. We employ a Taylor expansion about $x = 0$. Our choice of parameters enables us to write $z_n = r(0) + D_1 r(0) + \frac{1}{2} D_{11} r(\lambda z_1)$ for some $0 < \lambda < 1$. But $r(0) = 0$ and $D_1 r(0) = 0$ since H is tangent to K there. Hence $z_n = \frac{1}{2} D_{11} r(\lambda z_1) < \alpha/2$. On the other hand, the height of $T(x,\delta)$ above H at $(z_1, 0, \cdots, 0)$ is $\delta - \sqrt{\delta^2 - z_1^2}$. But $\alpha/2 < \delta - \sqrt{\delta^2 - z_1^2}$ whenever $\delta < z_1^2/\alpha + \alpha/4$ which is true by our choice of δ. Thus $z \notin T(x,\delta)$ for any $z \in W \sim \{x\}$.

Now choose a hyperplane H' parallel to H so that the boundary of K between H and H' lies entirely in W. This is possible since W contains no line segments. Let B' be the subset of B between H and H'. Then $B' \subseteq K$ and, moreover, for some $\varepsilon > 0$, $T(x,\varepsilon) \subseteq B'$. Thus $x \in L_\varepsilon$. Since x was chosen arbitrarily, the same result follows for every point in the relative interior of W (though, perhaps, the ε may vary) and the result then follows from (6.1).

8. Remarks

While our proofs have really dealt with support functions, they may be phrased without any reference to support functions at all. Simply take the constructions on support functions given at each stage and carry out the parallel constructions on the body K. Such a procedure would enable us to eliminate most of Section 2. I chose to leave the proof as it is because the arguments seem much more natural this way and because decomposing functions seems to be somewhat easier than decomposing sets.

There is an unsolved problem which arises rather naturally from the discussion above. Let $L = \bigcap \{L_\varepsilon : \varepsilon > 0\}$. Then $K \sim L$ represents the set of all "vertices", in the sense that these are points which admit no tangent ball of positive radius from the interior of K. It is easy to see that L is dense in the boundary of K and too difficult to find examples showing that $K \sim L$ is dense in the boundary of K. The question remains: Is the surface measure of $K \sim L$ always 0?

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